Bell's Inequality – Reality and Locality

Abstract

We will derive a simple version of the Bell's Inequality and use playing cards to illustrate how and why the Inequality is obeyed by classical systems. It is then pointed out some quantum systems can violate the Bell's Inequality, and we will use our classical example to speculate the implications of such violation, namely that some classical concepts we take for granted are in fact questionable, and that classical realism and locality could not both be correct when quantum mechanics is involved.

Introduction

Einstein once asked the question: If nobody is looking at the moon, does it still exist? Despite what you might consider to be the obvious answer, the truth is far subtler.

Let's say you come across a playing card facing down, without flipping the card over, you would have to conclude that the card has ½ probability being red in color, and ½ probability being black. In other words, $P(B)=1/2$ and $P(R)=1/2$. You resort to using probability merely because of your ignorance of the state of the card, even though you firmly believe the card must be either black or red. Classically, even if nobody has looked at the card, the color of the card is already determined and is what we will call the classical reality below.

Quantum mechanically, things could be quite different. Although actual playing cards are classical system, we could imagine a quantum version of the playing cards for which quantum behaviors remains prominent. (In practice one may think of our "quantum cards" as the spin of an electron, which has two possible states, up or down.) A quantum playing card could, for instance, be in a state $B+R$, which gives the same probability of $P(B)=1/2$ and $P(R)=1/2$ should you choose to flip the card over. However, as long as nobody flips the card over, the state of the card remains $B+R$, which for lack of a better description, can be said to be both black and red at the same time. As you flip the card over, the state of the card changes from $B+R$ to either black or red. In quantum mechanics we believe $B+R$ is already the most complete description of the card before the flip and has nothing to do with the ignorance of the observer. Even if, say, you find the card to be black after the flip, you cannot use this observation to infer that the state of the card was B before the flip. According to the orthodox view, the act of observation "collapses" the wave function to the state that you observe.

Table 1: The state of the card before and after the flip.

	Classical	Quantum
Before flip	B or R	$B+R$
After flip	B or R	B or R

In summary, in classical physics, the color of the card is fixed even before the observation is made, the reality of the color of the card exists even if the color is unknown to the observer. In quantum mechanics, the color was indefinite before the flip, and only becomes definite after the observation. In quantum mechanics, there is nothing to be gained to ask what the color of the card *really* is without actually flipping the card. There is no underlying reality beyond the uncertain state $B + R$.

Such ideas may seem rather abstract and philosophical, but somewhat surprisingly, there is actually a way to experimentally test which version of reality is correct based on the Bell's Inequality. Due to our limited budget and equipment, we will not be able to reproduce the quantum side of the experiment, however, it is relatively easy to reproduce the logic on the classical side.

Suppose you have multiple cards facing down, and we will assume each card has to be either black or red even if you are not looking. We will show that this "assumption" of classical reality already put a constraint on the probability of the outcome. Quantum mechanical systems, as it turns out, can violate this constraint (violate Bell's Inequality), and may be used to infer that the quantum versions of the playing cards do not have pre-determined colors before they were flipped over.

Quantum Mechanics Result

Before we describe our classical experiment, you need to understand the quantum version of what we will try to imitate in this lab. The setup is show in Figure 1.

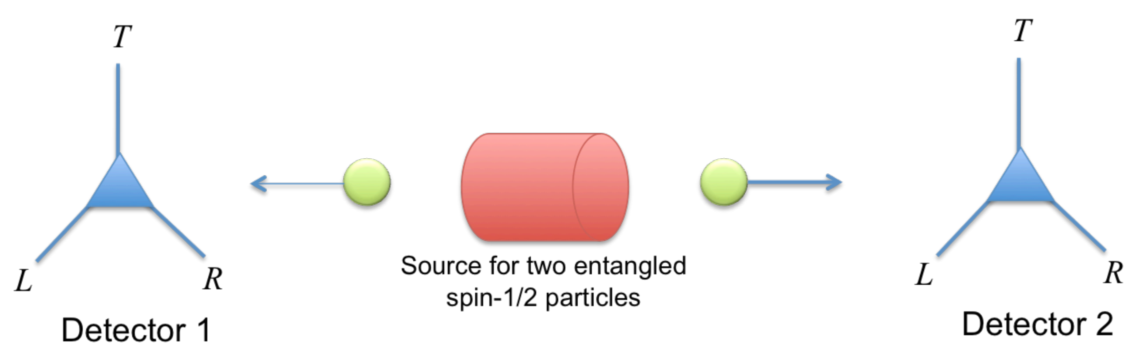

Figure 1: Schematic of a spin measuring experiment on two entangled particles.

Each of the particles traveling to the detectors can be either spin-up or spin down, and each detector can have one of three settings: T, L, R. No matter the setting, the detector will always give an answer of either *up* or *down*, corresponding to the spin in the direction of the setting. For example, if the detector 1 reads *down* when it is set to L, it means the spin of the particle entering the detector is opposite to L in Figure 1.

It turns out it is possible to produce entangled (or correlated) particles such that the two detectors always give the same reading[1] as long as they are in the same setting. For example, if Detector 1 (in setting L) measures *down*, then Detector 2 (in setting

[1] The actual quantum system actually gives exactly the opposite reading, but we will simplify the discussion by saying the readings are the same.

L) must also give a measurement of *down*. This in itself is not difficult to explain classically. One could imagine, for example, that both of the particles were spin-*down* in the *L* direction when they were produced (even before they were observed or measured by the detectors). The act of measurement simply uncovers the fact that they were produced in a correlated manner, in other words, both particles has pre-determined answers (or pre-determined reality) to whatever settings they encounter at the detectors. You may think of each particle carrying a cheat sheet, telling it what to say to the detector in different setting. As long as their pre-determined answers (or cheat sheets) match one another, the two detectors will always get identical answers when their settings are identical.

What is interesting is when the settings of the two detectors are allowed to be different. Now, it is possible to getting different answers in the two detectors. For example, if Detector 1 (in setting *L*) measures *down*, Detector 2 (in setting *T*) could give either *up* or *down*. The interesting fact found experimentally for certain quantum entangled systems is that you will be just as likely get the same answers from the two detectors as you do the opposite, but it turns out such an observation is not possible if the particles carry pre-determined sets of answers. Classically, it turns out it is always more likely for the detectors to get the same measurements, and this is what we will try to reproduce below.

Part A. Classical Reality with Locality

In this part, we will study the classical version of a system that obeys Bell's Inequality. Locality is ensured by the fact that what happens at one pile will not affect the cards at the other pile.

Procedure

1. Place three cards face down on the table. We will call the cards "*Top*", "*Left*" and "*Right*", or simply *T, L, R.*

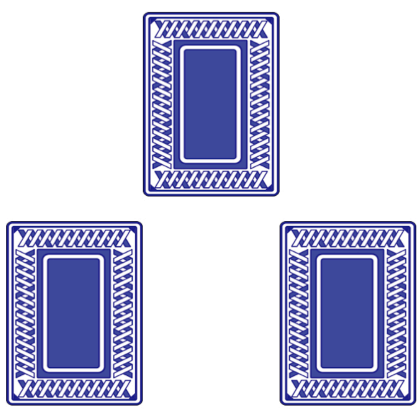

Figure 2: Three cards face down

2. Have a student (the designated "peeper") from your group peeks at the cards and note which ones are black, which ones are red (without telling anyone else). You do not need to note anything beyond the color of the cards.

3. For example, if you see the cards as in Figure 3, then it is *Top =B, Left =B, Right =R*, or simply *BBR*. Of course your own cards could be different from the ones in this example.

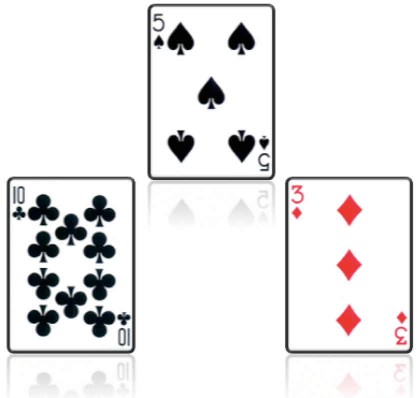

Figure 3: Example of BBR in the original pile

4. The student who knows the colors of the cards now picks three more cards from the deck with the same colors (e.g. BBR if you have the card configuration in Figure 3) and places them *face down* on the opposite side of the room from the original three. Make sure the colors correctly match the original three card by card and that no one except the designated peeper knows the colors of the cards. The colors of the cards are now exactly correlated.
5. Have one student to stand next to each pile and each throw a dice. Turn over one card in each pile according to the value of the dice (1, 2 = T; 3, 4 = L; 5, 6 = R), and record the color of the card in Table 2.
6. Get the average scores from the other groups and complete the class average in Table 3. Is your class average greater than 1/2?

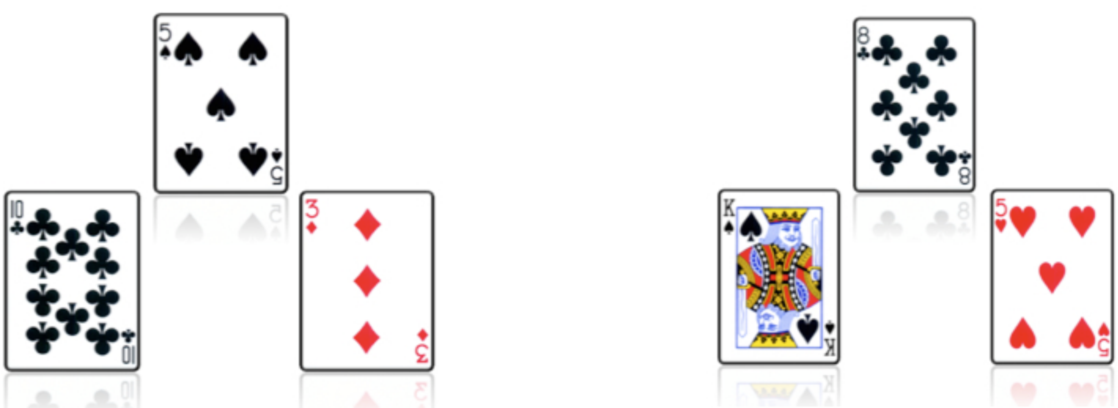

Figure 4: An example of the two piles with colors matching card by card. They should all be facing down so no one except the peeper has knowledge of the cards.

Table 2: Colors of the cards

Trial	Pile 1		Pile 2		Score (Colors the same? Yes=1/No=0)
	Card chosen (T, L or R)	Color (B or R)	Card chosen (T, L or R)	Color (B or R)	
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
				Average Score	

Table 3: Scores from the other groups and class average

Group	1 (You)	2	3	4	5	6	Class Average
Scores							

Discussion

Bell's Inequality states that the average score should be larger than 1/2 if the sample size is sufficiently large (see Appendix A). The underlying assumptions relevant to our discussion here are:

i. The colors of the cards exist even if no one flips the cards over. This is the assumption of classical reality.
ii. There are no interactions between the two piles once they are separated. In other words, each pile remains undisturbed by what happens at the other pile. This is the assumption of locality, that a system cannot be influenced by another system a long distance away without the transmission of signals or information between the two systems.

When quantum mechanically entangled systems (not the playing cards here!) were experimentally found to violate the Bell's Inequality (have a score less than or equal to[2] 1/2), it implied one of the two assumptions above were invalid. Most physicists abandon classical realism in favor of locality, the second option is to abandon locality in favor of classical realism. If you choose the first option, then you will have to accept the notion that a card does not have a definite color if it is not flipped over. If you choose the second, then you have to accept there are non-local influences that can somehow influence two systems faster than light could travel. In Part B below, we will, albeit much less accurately, attempt to describe non-local influences.

Part B. Classical Reality with Non-locality (Optional)

In the part we will construct a blatantly unrealistic and inaccurate model of a non-local influence. It merely serves to illustrate that when one allows non-local influences, Bell's Inequality could be violated.

Procedure

1. Have the designated peeper up the two piles of cards as before.
2. Have one of the students at the two piles flip over a card first. The designated peeper needs to observe the position (Top, Left or Right) of the flipped card.
3. The designated peeper now goes to the other pile (before any card is flipped) and replaces the two cards that is not the counterpart of the flipped card so that they both have the opposite color than the flipped card. For example, if the Top card is flipped in pile 2 and it is black, then the peeper needs to replace the Left and Right card so that they are both red.
4. Now the student at the second pile can flip over a card chosen at random and record the result in Table 4.
5. Get the average scores from the other groups and complete the class average in Table 5. Is your class average greater than 1/2?

Table 4: Colors of the cards when there are non-local influences.

Trial	Pile 1		Pile 2		Score (Colors the same? Yes=1/No=0)
	Card chosen (T, L or R)	Color (B or R)	Card chosen (T, L or R)	Color (B or R)	
1					
2					
3					
4					
5					
6					
				Average Score	

[2] Actually the quantum mechanical version (using spins instead of color) should have a score of exactly 1/2.

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Table 5: Scores from the other groups and class average

Group	1 (You)	2	3	4	5	6	Class Average
Scores							

Discussion

The action of the peeper represents an influence (or a signal) that travels to the second pile. In the presence of such effect, the score can be arranged to violate Bell’s Inequality (so that $\langle s \rangle \leq 1/2$). However, since the peeper only has a finite speed, it takes a finite amount of time for him to walk from pile 1 to pile 2. If the student at pile 2 decides to flip over a card before the peeper arrives, then this whole setup would fail. For the peeper to truly facilitate a violation of the Bell’s Inequality even if the students flip the cards simultaneous, he must somehow be able to exert (non-local) influence at a distance simultaneously as well.

In quantum mechanics, the observation on one particle in one location is supposed to instantaneously “collapse” the wave function of an entangled particle a long distance away, although no one is quite sure how. In particular you may remember that the concept of simultaneity is relative (two events could occur at the same time to one observer but occur at a different time to another), so how such influence could be “simultaneous” also demands further understanding.

Questions

Part A

Exercise A.1:

What is the total number of possible color configurations for three cards? You can answer this by filling the table below and count the number of configurations you come up with. The first two columns have been filled, complete the other columns by filling in either B or R. You will not need all the columns so leave the unused columns blank.

Table 6: Counting color configurations

Count	1	2	3	4	5	6	7	8	9	10	11
Top	B	B									
Left	B	B									
Right	B	R									

Exercise A.2:

What is the total number of ways to flip open the cards at the two piles? Again you can do so by filling and counting the table below with T, L or R. For example, if the student at pile 1 flipped open the top card, and the student at pile 1 flipped the left card, it will be recorded as T and L below (as in column 2). You will not need all the columns so leave the unused columns blank.

Table 7: Counting the number of ways of flipping over the cards

Count	1	2	3	4	5	6	7	8	9	10	11
Pile 1	T	T									
Pile 2	T	L									

Part B

Exercise B.1:

What is the probability of getting the same color, what is the probability of getting opposite colors? What is the expectation value of the score in part B? In other words, calculate $P(1)$, $P(0)$ and $\langle s \rangle$.

Appendix A: Calculating Probability (Optional)

In exercise 1 and 2, you should be able to show that there are 8 total card configurations, and 9 possible ways to open the cards.

Out of the 8 color configurations, 2 configurations (BBB and RRR) have all the cards carrying the same color, the other 6 configurations has two cards carrying the same card and one card different.

When the cards all carry the same color, no matter which of the 9 ways the students can choose to flip the cards, they will always get the same color[3]:

$P(\text{score is } 1 \mid \text{all cards same color}) = 1$.

When one out of the three cards have a different color, then out of the 9 ways the students can choose to flip the cards, 5 ways will give them the same color with each other:

$P(\text{score is } 1 \mid \text{one card different color}) = 5/9$.

In summary, no matter how the cards were prepared[4], as long as the students have no prior knowledge of the cards and picks the cards randomly, they are always more likely to find their cards matching in color ($score = 1$):

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[3] $P(A|B)$ is the notation for conditional probability, i.e., the probability that A occurs given condition B is met.

[4] This includes the case when the cards where not randomly prepared. For example, even if the student who prepares the cards only prepares the configuration BBR and nothing else, the conclusion still holds.

P(score is 1) > 1/2.

A prediction can therefore be made about the average score:

$$\langle s \rangle = \sum_s sP(s) = 0 \cdot P(0) + 1 \cdot P(1) > 1/2$$

While the average score $\langle s \rangle$ is a statistical quantity and therefore subject to fluctuations, in theory as long as a sufficiently large number of trials is done, its value should be larger than 1/2. This is our simplified version of the famous Bell's Inequality that is obeyed by classical system but is violated by certain quantum mechanically entangled systems. Note the very small number of assumptions that went into its derivation, one of the "assumptions" is that the colors of the cards exist whether the cards were flipped or not. This may not appear to be an assumption at all, but is indeed a point about classical reality that quantum mechanics challenges.

What constitutes a *large enough* sample will be addressed in the **Appendix B**.

Appendix B: Determining the size of the sample (Optional)

In this appendix, we will estimate the sample size necessary to ensure a reasonable check on the Bell's Inequality. We will begin with an extra assumption that was not necessary in the derivation of the Bell's Inequality earlier, namely, we will assume the 8 color configurations to be equally probable. In other words, we assume the original three cards were picked from the deck randomly. It is not a necessary assumption, but will make out analysis of the sample size simpler. Under this assumption, we have:

P(score is 1)

= P(score is 1 |all cards same color)×P(all cards same color) + P(score is 1 |one card different color)×P(one card different color)

= 1×2/8 + 5/9×6/8

=2/3.

In summary, we have the following values for *P(s)*:

P(0) = 1/3, P(1) = 2/3.

The expectation and the standard deviation of the score is therefore:

$$\langle s \rangle = \sum_s sP(s) = 0 \cdot P(0) + 1 \cdot P(1) = 2/3$$

$$\sigma_1 = \sqrt{\langle s^2 \rangle - \langle s \rangle^2} = \sqrt{2/3 - (2/3)^2} = \frac{\sqrt{2}}{3} = 0.47$$

By the central limit theorem, if we average over N trials, the standard deviation will become:

$$\sigma_N = \frac{\sigma_1}{\sqrt{N}} = \frac{0.47}{\sqrt{N}}$$

If we want to measure $\langle s \rangle = 2/3 > 1/2$ as statistically significant, we will want $\langle s \rangle$ to be at least a standard deviation away from 1/2:

$$2/3 - 1/2 > \sigma_N \Rightarrow 0.17 > \frac{\sigma_1}{\sqrt{N}} \Rightarrow N > (\frac{0.47}{0.17})^2 = 7.64$$

Therefore if the total number of trial is above 7, the probability of experimentally getting $\langle s \rangle < 1/2$ should be less than 16 %. In other words, one would obtain the expected result, $\langle s \rangle > 1/2$ with a likelihood[5] of more than 84%.

If one desires to increase the likelihood further, one may demand $\langle s \rangle$ to be two standard deviations away from 1/2:

$$2/3 - 1/2 > 2\sigma_N \Rightarrow 0.17 > 2(\frac{\sigma_1}{\sqrt{N}}) \Rightarrow N > (2 \cdot \frac{0.47}{0.17})^2 = 30.6$$

This will increase the likelihood of obtaining $\langle s \rangle > 1/2$ to about 98%, although it involves a substantially higher number of trials.

[5] One should however be cautious because the approximation of central limit theorem is not strictly satisfied due to the small number of trials.